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Bose–Einstein condensation in an Einstein universe

Surjit Singh and R K Pathria
Guelph-Waterloo Program for Graduate Work in Physics, Waterloo Campus, University of Waterloo, Waterloo, Ontario, N2L 3G1, Canada

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Abstract. Taking into account the possibility of particle–antiparticle pair production, we have investigated the onset of Bose–Einstein condensation in an ideal relativistic Bose gas confined to an Einstein universe. Through an extensive use of the Poisson summation formula, we have carried out an explicit evaluation of the summations-over-states appearing in the problem, which enables us to derive rigorously the condensate fraction, ρ_0/ρ, and the specific heat at constant volume, c_ρ, of the system as *smooth* functions of temperature—from $T \gtrsim T_c$ down to $T = 0$ K. A detailed asymptotic analysis of the growth of the condensate and of the specific-heat maximum is then carried out and, in each case, finite-size corrections to the standard bulk results are obtained in explicit terms. Special cases of the non-relativistic and extreme relativistic versions of the model are examined at some length and, wherever possible, a comparison is made with the findings of the previous authors.

1. Introduction

A few years ago, using the formalism developed by Pathria and his collaborators for the study of Bose–Einstein condensation in finite systems (Pathria 1972a, Greenspoon and Pathria 1973, 1974, 1975, Zasada and Pathria 1976, 1977), Al'taie (1978) carried out a detailed investigation of the critical behaviour of an ideal Bose gas confined to the background geometry of an Einstein universe. The primary motivation for that investigation was to extend the study of Bose–Einstein condensation to curved space and to determine the manner in which the physical properties of the system are affected by the inherent finiteness of the space available. Owing to the spherical symmetry of the problem, the analysis in this case involves summations over one quantum number only and is, therefore, expected to be mathematically tractable. However, as Al'taie discovered in his investigation, the problem is tractable only in its non-relativistic version, with the result that he had to confine his analysis to this case alone.

Al'taie examined the twin problems of (i) the growth of the condensate, and (ii) the behaviour of the specific heat, especially the location and height of its maximum, as a function of temperature. He showed that, apart from the expected effect of 'smoothening out' the singularities of the thermodynamic functions of the system at the erstwhile critical point $T = T_c$, the finiteness of the system resulted in the enhancement of the condensate fraction, the displacement of the specific-heat maximum towards higher temperatures and the reduction in the height of the maximum—corrections to the bulk behaviour in each case being of order a^{-1}, where a denotes the radius of (the spatial part of) the Einstein universe.

Since the problem of Bose–Einstein condensation in the Einstein universe is inherently relativistic in nature, a more complete analysis of the situation is clearly in order. This was indeed attempted by Aragão de Carvalho and Goulart Rosa (1980), but only with partial success. In view of the aforementioned intractability of the finite-size terms in the relativistic case, these authors could not derive the desired results in closed form and were forced to draw only qualitative, or at best semi-quantitative, conclusions about the effects in question. Nevertheless, they did show quite generally that, for the study of finite-size effects in this problem, the neglect of relativistic effects was totally unjustified.

In this paper we carry out a rigorous analysis of this problem, including not only the conventional relativistic effects but also the possibility of particle–antiparticle pair production which has lately been shown to be an essential ingredient of the relativistic case; for details, see Haber and Weldon (1981, 1982) and Singh and Pandita (1983). A study of Bose–Einstein condensation in a curved space of finite size, including the above mentioned effects, appeared at first sight to be rather formidable. It turned out, however, that the inclusion of antiparticles into the problem rendered the analysis far more tractable than it originally was and enabled us to derive finite-size corrections to the relativistic bulk behaviour of the system in a closed form. We could, therefore, draw as rigorous and as thorough a set of conclusions in the relativistic case as Al'taie had done in the corresponding non-relativistic case. Remarkably enough, our findings are qualitatively similar to those of Al'taie's; quantitatively, of course, they depend on the severity of the relativistic effects which, in turn, is determined by the parameter ρ/m^3, where ρ is (essentially) the 'number density' in the system while m is the particle mass.

For simplicity, we confine our study to the case of spinless particles ($s=0$). The case of spin-1 particles can be studied in a similar fashion.

2. Formulation of the problem

We consider an ideal Bose gas composed of N_1 particles and N_2 antiparticles, each of mass m, confined to an Einstein universe of (spatial) radius a. Since particles and antiparticles are supposed to be created in pairs, the system is governed by the conservation of the number Q $(=N_1-N_2)$, rather than of the numbers N_1 and N_2 separately; the conserved quantity Q may be looked upon as a kind of generalised 'charge'. In equilibrium, the chemical potentials of the two species will be equal and opposite: $\mu_1=-\mu_2=\mu$, say, with the result that (Haber and Weldon 1981)

$$N_1=\sum_\varepsilon [e^{\beta(\varepsilon-\mu)}-1]^{-1}, \qquad N_2=\sum_\varepsilon [e^{\beta(\varepsilon+\mu)}-1]^{-1}, \tag{1}$$

where $\beta=1/T$ and $\varepsilon=(k^2+m^2)^{1/2}$; for economy, we shall use units such that $\hbar=c=k_B=1$. Note that both ε and μ include the rest energy m of the particle (or the antiparticle) and, for the mean occupation numbers in the various states to be positive definite, we must have: $|\mu|\leq m$. Assuming that, to begin with, $\mu>0$, it readily follows that $N_1>N_2$ and hence $Q>0$. In view of the conservation of Q, μ then stays positive under all circumstances. Without loss of generality, we shall assume that this indeed is the case.

The eigenvalues, k_n, of the wavenumber k for a free particle confined to the Einstein universe are given by (see, for example, Schrödinger 1938)

$$k_n=n/a \qquad (n=1,2,3,\ldots), \tag{2}$$

with multiplicity $g_n = n^2$. The 'charge density' ρ is then given by the expression

$$\rho \equiv \frac{N_1 - N_2}{V} = \frac{2}{V} \sum_{j=1}^{\infty} \sinh(j\beta\mu) \sum_{n=1}^{\infty} n^2 \exp\left[-j\beta m \left(1 + \frac{n^2}{m^2 a^2} \right)^{1/2} \right]. \tag{3}$$

Applying Poisson's summation formula,

$$\sum_{n=1}^{\infty} f(n) = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} f(n) = \tfrac{1}{2} \sum_{q=-\infty}^{\infty} \mathscr{F}(q), \tag{4}$$

where

$$\mathscr{F}(q) = \int_{-\infty}^{\infty} f(n)\, \mathrm{e}^{2\pi i q n}\, \mathrm{d}n, \tag{5}$$

and substituting $V = 2\pi^2 a^3$ (Pathria 1974), equation (3) takes the form

$$\rho = \frac{m^2}{\pi^2 \beta} \sum_{j=1}^{\infty} j \sinh(j\beta\mu) \sum_{q=-\infty}^{\infty} \left(\frac{K_2(\beta m z)}{z^2} - (\beta m q'^2) \frac{K_3(\beta m z)}{z^3} \right), \tag{6}$$

where $K_\nu(z)$ are the modified Bessel functions while

$$z = \sqrt{j^2 + q'^2}, \qquad q' = (2\pi a/\beta) q. \tag{7}$$

The term with $q = 0$ corresponds to the bulk result, see (4) and (5),

$$\rho_{\mathrm{B}}(\beta, \mu) = \frac{m^3}{\pi^2} \sum_{j=1}^{\infty} (j\beta m)^{-1} \sinh(j\beta\mu) K_2(j\beta m), \tag{8}$$

which agrees with the corresponding result obtained by Singh and Pandita (1983). We may, therefore, write

$$\rho = \rho_{\mathrm{B}}(\beta, \mu) + \frac{2m^2}{\pi^2 \beta} \sum_{q=1}^{\infty} \sum_{j=1}^{\infty} j \sinh(j\beta\mu) \left(\frac{K_2(\beta m z)}{z^2} - (\beta m q'^2) \frac{K_3(\beta m z)}{z^3} \right). \tag{9}$$

Clearly, the terms with $q \neq 0$ contain both the condensate and the finite-size effects in the problem.

The sumation over j appearing in (9) can also be handled with the help of the Poisson summation formula,

$$\sum_{j=1}^{\infty} f(j) = \tfrac{1}{2} \sum_{j=-\infty}^{\infty} f(j) = \tfrac{1}{2} \sum_{l=-\infty}^{\infty} \int_{-\infty}^{\infty} f(j) \cos(2\pi l j)\, \mathrm{d}j, \tag{10}$$

so that

$$\rho = \rho_{\mathrm{B}}(\beta, \mu) + \frac{2m^2}{\pi^2 \beta} \sum_{q=1}^{\infty} \sum_{l=-\infty}^{\infty} \int_0^{\infty} j \sinh(j\beta\mu') \left(\frac{K_2(\beta m z)}{z^2} - (\beta m q'^2) \frac{K_3(\beta m z)}{z^3} \right) \mathrm{d}j, \tag{11}$$

where

$$\mu' = \mu + (2\pi i/\beta) l, \tag{12}$$

while z and q' are given by (7). The integration over j is somewhat involved but can be carried out exactly; see appendix. Using (A5) and noting that (Gradshteyn and Ryzhik 1965)

$$K_{1/2}(z) - z K_{3/2}(z) = -z K_{-1/2}(z) = -(\tfrac{1}{2}\pi z)^{1/2}\, \mathrm{e}^{-z}, \tag{13}$$

we obtain after some algebra

$$\rho=\rho_{\rm B}(\beta,\mu)-\frac{1}{\pi\beta}\sum_{q=1}^{\infty}\sum_{l=-\infty}^{\infty}\mu'(m^2-\mu'^2)^{1/2}\exp[-2\pi aq(m^2-\mu'^2)^{1/2}]. \tag{14}$$

The summation over q is now straightforward, with the result that

$$\rho=\rho_{\rm B}(\beta,\mu)-\frac{1}{\pi\beta}\sum_{l=-\infty}^{\infty}\mu'(m^2-\mu'^2)^{1/2}\{\exp[2\pi a(m^2-\mu'^2)^{1/2}]-1\}^{-1}. \tag{15}$$

It may be emphasised here that in going from equation (3) to (15) we have made no approximation whatsoever.

Now, the onset of Bose–Einstein condensation is marked by the fact that the chemical potential μ of the system approaches the single-particle rest energy m. In that event, the bulk term appearing in (15), and given explicitly by (8), can be expressed as (Singh and Pandita 1983)

$$\rho_{\rm B}(\beta,\mu)=\rho_{\rm B}(\beta,m)-\frac{m}{2\pi\beta}(m^2-\mu^2)^{1/2}+{\rm O}(m^2-\mu^2). \tag{16}$$

At the same time, the term with $l=0$ in (15) dominates heavily over terms with $l\neq 0$. To see this, we recall (12) and write the summation over l as

$$\mu(m^2-\mu^2)^{1/2}\{\exp[2\pi a(m^2-\mu^2)^{1/2}]-1\}^{-1}$$
$$+\sum_{l=-\infty}^{\infty}{}'(\mu+2\pi{\rm i}\beta^{-1}l)[(m^2-\mu^2)+4\pi^2\beta^{-2}l^2-4\pi{\rm i}\mu\beta^{-1}l]^{1/2}$$
$$\times(\exp\{2\pi a[(m^2-\mu^2)+4\pi^2\beta^{-2}l^2-4\pi{\rm i}\mu\beta^{-1}l]^{1/2}\}-1)^{-1}, \tag{17}$$

where the primed summation implies that the term with $l=0$ is excluded from this sum. As $\mu\to m$, so long as $a(m^2-\mu^2)^{1/2}$ is of order unity or less, the main term in (17) would make a significant contribution to ρ. The other terms, however, are *at best* of order $\exp[-a(m/\beta)^{1/2}]$, i.e. ${\rm O}({\rm e}^{-a/\lambda_{\rm T}})$, where $\lambda_{\rm T}$ $(=(2\pi\beta/m)^{1/2})$ denotes the mean thermal wavelength of the particles. Making the very reasonable assumption that $a\gg\lambda_{\rm T}$, these terms can be dropped with impunity. The important thing to note is that no errors of order $(\lambda_{\rm T}/a)^n$ are committed if we retain only the term with $l=0$.

Introducing the *thermogeometric parameter* y (Pathria 1983) appropriate to this problem, namely

$$y=\pi(m^2-\mu^2)^{1/2}a, \tag{18}$$

equation (15) may now be written as

$$\rho=\rho_{\rm B}(\beta,\mu)-\frac{\mu}{\pi^2\beta a}\frac{y}{{\rm e}^{2y}-1}. \tag{19}$$

Combining this with (16), we finally obtain

$$\rho=\rho_{\rm B}(\beta,m)-(m/2\pi^2\beta a)y\coth y+{\rm O}(y^2/a^2). \tag{20}$$

Equation (20) constitutes our basic result from which the crucial parameter $y(\rho,\beta)$ can be determined.

At this point we would like to make two observations on our passage from equation (15) to (20). First, one should note how agreeably the singular term, containing $(m^2-\mu^2)^{1/2}$, in the expansion (16) of the bulk function $\rho_{\rm B}(\beta,\mu)$ is cancelled by a similar term residing in the function $y/(e^{2y}-1)$, to yield a final result which is a function of y^2 only and, hence, is non-singular. For a finite system, this should indeed be the case. Second, the practical limits of accuracy to which the various physical properties of the system may now be calculated are determined by the accuracy of the expansion of the bulk term $\rho_B(\beta, \mu)$ and *not* by the term containing finite-size effects (which has been obtained in a closed form). For a study of finite-size effects to order a^{-1}, it is sufficient to consider terms displayed explicitly in equation (20).

3. Growth of the condensate

The condensate in this problem arises from the single-particle ground state $n=1$ and is given by the expression, see equations (1)–(3),

$$\rho_0=(1/2\pi^2a^3)\{[\exp\beta(\varepsilon_1-\mu)-1]^{-1}-[\exp\beta(\varepsilon_1+\mu)-1]^{-1}\}. \tag{21}$$

In view of the fact that, for $ma \gg 1$,

$$\varepsilon_1 \simeq m(1+1/2m^2a^2) \tag{22}$$

and, for μ approaching m,

$$\mu \simeq m(1-y^2/2\pi^2m^2a^2), \tag{23}$$

the expression for ρ_0 effectively reduces to

$$\rho_0 \simeq 1/2\pi^2a^3\beta(\varepsilon_1-\mu) \simeq m/a\beta(y^2+\pi^2). \tag{24}$$

Equations (22)–(24) show that, so long as $y^2=O(1)$, the condensate density in the system remains of order a^{-1}, which means that Q_0 remains of order a^2, i.e. $O(Q^{2/3})$. A macroscopic growth of the condensate results only when $y^2 \to -\pi^2$ or, in other words, $\mu \to \varepsilon_1$.

For a detailed study of this process, we make use of the identity

$$\begin{aligned} y\coth y &= 1+2y^2\sum_{q=1}^{\infty}\frac{1}{y^2+\pi^2q^2} \\ &= -\frac{2\pi^2}{y^2+\pi^2}+3+2y^2\sum_{q=2}^{\infty}\frac{1}{y^2+\pi^2q^2}, \end{aligned} \tag{25}$$

whereby equations (20) and (24) yield, to leading order in a^{-1},

$$\rho_0=[\rho-\rho_B(\beta, m)]+\frac{m}{\pi^2\beta a}\left(\tfrac{3}{2}+y^2\sum_{q=2}^{\infty}\frac{1}{y^2+\pi^2q^2}\right). \tag{26}$$

For comparison we note that, in the case of the bulk system ($a\to\infty$), there exists a critical temperature, $\beta=\beta_c$, given by

$$\rho_B(\beta_c, m)=\rho. \tag{27}$$

Using the function $W(\beta, \mu)$ of Singh and Pandita (1983), namely

$$W(\beta, \mu)=2\sum_{j=1}^{\infty}(j\beta m)^{-1}\sinh(j\beta\mu)K_2(j\beta m), \tag{28}$$

this condition may be written as, see equation (8),

$$W(\beta_c, m) \equiv 2 \sum_{j=1}^{\infty} (j\beta_c m)^{-1} \sinh(j\beta_c m) K_2(j\beta_c m) = \frac{2\pi^2 \rho}{m^3}. \tag{29}$$

The bulk condensate density is then given by the *singular* expression

$$(\rho_0)_B = \begin{cases} 0 & (\beta < \beta_c) \quad (30a) \\ \rho\left(1 - \dfrac{W(\beta, m)}{W(\beta_c, m)}\right) & (\beta > \beta_c). \quad (30b) \end{cases}$$

In the case of the finite system, on the other hand, the condensate density is given by the non-singular expression

$$\rho_0 = \rho\left(1 - \frac{W(\beta, m)}{W(\beta_c, m)}\right) + \frac{m}{\pi^2 \beta a}\left(\tfrac{3}{2} + y^2 \sum_{q=2}^{\infty} \frac{1}{y^2 + \pi^2 q^2}\right), \tag{31}$$

where y^2, as a function of β, is determined by, see equation (20),

$$y \coth y = -\frac{2\pi^2 \beta a \rho}{m}\left(1 - \frac{W(\beta, m)}{W(\beta_c, m)}\right). \tag{32}$$

Note that the parameter β_c denotes the critical temperature of the bulk system; while no such parameter exists in the case of the finite system, it still serves as a useful reference point in this case.

For a quantitative comparison of equations (30) and (31), we must solve (32) for $y^2(\beta)$, which can only be done numerically. A few general results, however, can be obtained as follows. First of all, at $\beta = \beta_c$, we have: $(y \coth y)_c = 0$, i.e. $y_c^2 = -\frac{1}{4}\pi^2$. Equation (31) then gives, see also (24),

$$(\rho_0)_c = 4m/3\pi^2 \beta_c a \qquad (T = T_c), \tag{33}$$

which implies that $(Q_0)_c = O(Q^{2/3})$. For $y^2 \gg 1$, equation (32) reduces to

$$y \simeq (\beta_c m)^2 |\mathrm{d}W/\mathrm{d}\beta|_{\beta=\beta_c} at \qquad (0 < t \ll 1, at \gg 1), \tag{34}$$

where $W \equiv W(\beta, m)$, while t is the dimensionless temperature variable

$$t = (T - T_c)/T_c \simeq (\beta_c - \beta)/\beta_c. \tag{35}$$

One then obtains, directly from (24),

$$\rho_0 \simeq 1/\beta_c^5 m^3 [(\mathrm{d}W/\mathrm{d}\beta)_c]^2 a^3 t^2, \tag{36}$$

which implies that $Q_0 = O(1)$. At lower temperatures ($\beta > \beta_c$), we run into the limiting situation $y^2 \to -\pi^2$, i.e. $y \coth y \to -\infty$. This requires that $|at| \gg 1$, i.e. $|t|$ be of order greater than a^{-1} (which includes the possibility that $|t|$ may even be O(1)). Equation (31) now gives

$$\rho_0 \simeq (\rho_0)_B + 3m/4\pi^2 \beta a, \tag{37}$$

which holds all the way down to 0 K.

Comparing equations (33) and (36) with (30*a*), and (37) with (30*b*), we conclude that, *irrespective of the relativistic effects in the problem*, the finiteness of the system causes an enhancement in the condensate density ρ_0 at *all* temperatures. For a considerable range of temperatures, covered by (37), the enhancement is directly proportional to T and inversely proportional to a. Around $T = T_c$, where $|t| = O(a^{-1})$, the enhancement is still inversely proportional to a but depends in a more complicated manner on T. For $T > T_c$, such that $(at) \gg 1$, the enhancement tapers off as a^{-3}. In qualitative terms, these findings are similar to the ones arrived at by Al'taie in the non-relativistic case. Quantitatively, too, we find complete agreement with Al'taie's results if the function $W(\beta, m)$ is replaced by its non-relativistic limit, namely

$$[W(\beta, m)]_{NR} = (\pi/2m^3)^{1/2}\zeta(\tfrac{3}{2})\beta^{-3/2} \qquad (\rho \ll m^3). \tag{38}$$

In the extreme relativistic limit, on the other hand, this function takes the form

$$[W(\beta, m)]_{ER} = (2\pi^2/3m^2)\beta^{-2} \qquad (\rho \gg m^3). \tag{39}$$

Using (38) or (39) in the bulk term of our basic equation (20), or in any subsequent result, the two limiting versions of the problem can be worked out in complete detail.

4. Derivation of the specific heat

For the subsequent analysis of the problem, we start with the expression for the energy density of the system, namely

$$\begin{aligned} u &= \frac{1}{2\pi^2 a^3}\sum_n n^2\varepsilon_n\{[\exp\beta(\varepsilon_n - \mu) - 1]^{-1} + [\exp\beta(\varepsilon_n + \mu) - 1]^{-1}\} \\ &= \frac{m}{\pi^2 a^3}\sum_{j=1}^{\infty}\cosh(j\beta\mu)\sum_{n=1}^{\infty} n^2\left(1 + \frac{n^2}{m^2a^2}\right)^{1/2}\exp\left[-j\beta m\left(1 + \frac{n^2}{m^2a^2}\right)^{1/2}\right]. \end{aligned} \tag{40}$$

Applying the Poisson summation formula (4), we obtain

$$\begin{aligned} u = \frac{m^2}{\pi^2\beta^2}\sum_{j=1}^{\infty}\cosh(j\beta\mu)\sum_{q=-\infty}^{\infty}\Bigg[&\left(-\frac{K_2(\beta mz)}{z^2} + (\beta mj^2)\frac{K_3(\beta mz)}{z^3}\right) \\ &-(\beta mq'^2)\left(-\frac{K_3(\beta mz)}{z^3} + (\beta mj^2)\frac{K_4(\beta mz)}{z^4}\right)\Bigg], \end{aligned} \tag{41}$$

where z and q' are the same as given in equation (7). First of all, we extract the bulk term (with $q = 0$), namely

$$u_B(\beta, \mu) = \frac{m^2}{\pi^2\beta^2}\sum_{j=1}^{\infty}\cosh(j\beta\mu)\left(\frac{-K_2(j\beta m)}{j^2} + \beta m\frac{K_3(j\beta m)}{j}\right). \tag{42}$$

Among the remaining terms (with $q \neq 0$), we convert the summation over j into a summation over l by using the Poisson summation formula (10)—modified for the fact that $f(j = 0)$ in the present calculation is non-vanishing. Using equation (A7) of the appendix and the recurrence relation (13), we obtain

$$\begin{aligned} u = u_B(\beta, \mu) &- \frac{m^2}{4\pi^4 a^2}\sum_{q=1}^{\infty}\left(-\frac{K_2(2\pi maq)}{q^2} + (2\pi ma)\frac{K_3(2\pi maq)}{q}\right) \\ &-\frac{1}{\pi\beta}\sum_{q=1}^{\infty}\sum_{l=-\infty}^{\infty}\mu'^2(m^2 - \mu'^2)^{1/2}\exp[-2\pi aq(m^2 - \mu'^2)^{1/2}], \end{aligned} \tag{43}$$

where μ' is the same as given in equation (12). Finally, neglecting terms $O(e^{-a/\lambda_T})$ or

$O(e^{-a/\lambda_C})$, where λ_T is the mean thermal wavelength and λ_C $(=m^{-1})$ the Compton wavelength of the particles, and carrying out the summation over q, we obtain the remarkably simple result

$$u = u_B(\beta, \mu) - \frac{\mu^2}{\pi^2 \beta a} \frac{y}{e^{2y}-1}, \tag{44}$$

where y is the same as given in (18). Combining (44) with (19), we obtain for the *thermal* energy density of the system

$$\bar{u} \equiv (u - m\rho) = \bar{u}_B(\beta, \mu) + \frac{\mu(m-\mu)}{\pi^2 \beta a} \frac{y}{e^{2y}-1}, \tag{45}$$

where

$$\bar{u}_B(\beta, \mu) \equiv [u_B(\beta, \mu) - m\rho_B(\beta, \mu)] = \frac{m^4}{2\pi^2} Z(\beta, \mu), \tag{46}$$

with

$$Z(\beta, \mu) = 2 \sum_{j=1}^{\infty} \left[\cosh(j\beta\mu) \left(\frac{K_3(j\beta m)}{(j\beta m)} - \frac{K_2(j\beta m)}{(j\beta m)^2} \right) - \sinh(j\beta\mu) \frac{K_2(j\beta m)}{(j\beta m)} \right]. \tag{47}$$

As $\mu \to m$, the function $Z(\beta, \mu)$ can be expressed as

$$Z(\beta, \mu) = Z(\beta, m) - \left(\frac{\partial Z}{\partial \mu} \right)_{\mu=m} (m-\mu) + \frac{2^{1/2}\pi}{\beta m^{5/2}} (m-\mu)^{3/2} + O(m-\mu)^2. \tag{48}$$

Noting that

$$\left(\frac{\partial Z(\beta, \mu)}{\partial \mu} \right)_{\mu=m} = -\frac{\beta}{m} \left(\frac{dW}{d\beta} \right),$$

where $W\,[\equiv W(\beta, m)]$ is given by (28), equations (45)–(48) lead to the final expression

$$\bar{u} = \frac{m^4}{2\pi^2} Z(\beta, m) + \frac{\beta m^2}{4\pi^4 a^2} \left(\frac{dW}{d\beta} \right) y^2 + \frac{1}{4\pi^4 \beta a^3} (y^3 \coth y) + O\left(\frac{y^4}{a^4} \right). \tag{49}$$

Note, once again, the cancellation of the singular terms and the emergence of a final expression which is a function of y^2 only.

To determine the specific heat of the system at constant volume, we need to know the quantity $(\partial y/\partial \beta)_\rho$. This can be obtained from the equation (see (8), (20) and (28))

$$\rho = \frac{m^3}{2\pi^2} W(\beta, m) - \frac{m}{2\pi^2 \beta d} (y \coth y) + O\left(\frac{y^2}{a^2} \right). \tag{50}$$

To the leading order in a,

$$\left(\frac{\partial y}{\partial \beta} \right)_\rho = \frac{\beta m^2 (dW/d\beta) a}{\coth y - y \operatorname{cosech}^2 y}, \tag{51}$$

whence

$$\begin{aligned} c_\rho &\equiv -\beta^2 (\partial \bar{u}/\partial \beta)_\rho \\ &= \frac{m^4 \beta^2}{2\pi^2} \left| \frac{dZ}{d\beta} \right| - \frac{(\beta m)^4 (dW/d\beta)^2}{2\pi^4 (\coth y - y \operatorname{cosech}^2 y)} \frac{y}{a}, \end{aligned} \tag{52}$$

where $Z \equiv Z(\beta, m)$. Equation (52) will enable us to study the leading finite-size correction to the bulk specific heat of the system—in particular, the rounding-off of the bulk singularity into a smooth maximum.

5. Location and height of the specific-heat maximum

First of all we examine the bulk situation in the vicinity of the critical point ($t=0$). For $t>0$ and $a\to\infty$, the thermogeometric parameter y is given by the asymptotic expression (34). Equation (52) then becomes

$$c_\rho(t>0)=\frac{m^4\beta_c^2}{2\pi^2}\left[\left|\frac{dZ}{d\beta}\right|_c+\left(\beta_c\left(\frac{d^2Z}{d\beta^2}\right)_c-2\left|\frac{dZ}{d\beta}\right|_c\right)t-\frac{\beta_c^4m^2}{\pi^2}\left|\frac{dW}{d\beta}\right|_c^3 t\right]. \tag{53}$$

For $t<0$ and $a\to\infty$, $y\to i\pi$. Setting $y=i(\pi-\varepsilon)$, where $0<\varepsilon\ll 1$, equation (32) gives

$$\varepsilon=\pi/(\beta_c m)^2|dW/d\beta|_c a|t|\to 0, \tag{54}$$

whence

$$c_\rho(t<0)=\frac{m^4\beta_c^2}{2\pi^2}\left\{\left|\frac{dZ}{d\beta}\right|_c+\left[\beta_c\left(\frac{d^2Z}{d\beta^2}\right)_c-2\left|\frac{dZ}{d\beta}\right|_c\right]t\right\}. \tag{55}$$

Thus, while c_ρ is continuous at the critical point, with

$$(c_\rho)_c=\frac{m^4\beta_c^2}{2\pi^2}\left|\frac{dZ}{d\beta}\right|_c, \tag{56}$$

its derivative with respect to temperature is discontinuous:

$$(\Delta c'_\rho)_c\equiv\left[\left(\frac{\partial c_\rho}{\partial t}\right)_{t=0-}-\left(\frac{\partial c_\rho}{\partial t}\right)_{t=0+}\right]=\frac{(\beta_c m)^6}{2\pi^4}\left|\frac{dW}{d\beta}\right|_c^3. \tag{57}$$

In the non-relativistic case

$$Z(\beta,m)=\tfrac{3}{2}\left(\frac{\pi}{2m^5}\right)^{1/2}\zeta(\tfrac{5}{2})\beta^{-5/2},\qquad W(\beta,m)=\left(\frac{\pi}{2m^3}\right)^{1/2}\zeta\left(\frac{3}{2}\right)\beta^{-3/2},$$
$$(\beta_c m)=\frac{1}{2\pi}\left[\zeta\left(\frac{3}{2}\right)\frac{m^3}{\rho}\right]^{2/3}, \tag{58}$$

which lead to the standard textbook results (see, for example, Pathria 1972b)

$$(c_\rho)_c=\frac{15}{4}\frac{\zeta(5/2)}{\zeta(3/2)}\rho,\qquad (\Delta c'_\rho)_c=\frac{27}{16\pi}\left[\zeta\left(\frac{3}{2}\right)\right]^2\rho. \tag{59}$$

In the extreme relativistic case

$$Z(\beta,m)=\frac{2\pi^4}{15m^4}\beta^{-4},\qquad W(\beta,m)=\frac{2\pi^2}{3m^2}\beta^{-2},\qquad (\beta_c m)=[m^3/3\rho]^{1/2}, \tag{60}$$

whence

$$(c_p)_c = \frac{4\pi^2}{5}\left(\frac{3\rho}{m^3}\right)^{1/2}\rho, \qquad (\Delta c'_p)_c = \frac{32\pi^2}{9}\left(\frac{3\rho}{m^3}\right)^{1/2}\rho, \tag{61}$$

in agreement with Haber and Weldon (1981) and with Singh and Pandita (1983).

In the case of finite a, this cusp-like singularity (at $t=0$) is rounded-off into a smooth maximum (at $t=t^*$). According to (52), the location of the maximum is determined by the condition

$$\frac{\coth y^* + y^* \operatorname{cosech}^2 y^* - 2y^{*2} \coth y^* \operatorname{cosech}^2 y^*}{(\coth y^* - y^* \operatorname{cosech}^2 y^*)^3} = \pi^2\left[\beta_c\left(\frac{d^2Z}{d\beta^2}\right)_c - 2\left|\frac{dZ}{d\beta}\right|_c\right] \Big/ \beta_c^2 m^2 \left|\frac{dW}{d\beta}\right|_c^3. \tag{62}$$

The right-hand side of this equation is seen to be O(1)—ranging from the value

$$\tfrac{10}{3}\pi\zeta(\tfrac{5}{2})/[\zeta(\tfrac{3}{2})]^3 = 0.788\ldots$$

in the non-relativistic limit to the value $\frac{27}{40} = 0.675$ in the extreme relativistic limit; the corresponding values of y^* turn out to be 1.887... and 1.090..., respectively. The location of the maximum is then given by

$$t^* = \frac{y^* \coth y^*}{(\beta_c m)^2 |dW/d\beta|_c}\frac{1}{a}, \tag{63}$$

which implies a shift, $O(a^{-1})$, towards higher temperatures. For the height of the maximum, we obtain

$$\frac{c_p^*}{(c_p)_c} = 1 + \left(\beta_c \frac{(d^2Z/d\beta^2)_c}{|dZ/d\beta|_c} - 2\right)t^* - \left(\frac{\beta_c^2 (dW/d\beta)_c^2}{\pi^2 |dZ/d\beta|_c}\right)\frac{y^*}{\coth y^* - y^* \operatorname{cosech}^2 y^*}\frac{1}{a}, \tag{64}$$

which implies a reduction in value, again $O(a^{-1})$. It is not difficult to see that the curvature of the specific-heat curve at $t=t^*$ is $O(a)$. It follows that, as a tends to infinity, the foregoing features of the system merge into a singularity at $t=0$.

We thus find that, quite generally, our results are qualitatively similar to the ones obtained by Al'taie in the non-relativistic case. In quantitative terms, there is complete agreement with Al'taie in the non-relativistic limit but significant differences appear when relativistic effects assume importance.

Finally we note that, in a recent review of the problem of relativistic Bose–Einstein condensation, Landsberg (1981) has made some intriguing speculations concerning Bose condensation of both massive and massless photons, leaving one with the conjecture that 'matter may be regarded as condensed black-body radiation'. It will clearly be of interest to see how our considerations in the present paper would affect those speculations. We hope to pursue this line of thought in a separate investigation.

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Appendix

We wish to evaluate the integral

$$I_\nu(x, y; \xi) = \int_0^\infty j \sinh(jx) \frac{K_\nu[y(j^2+\xi^2)^{1/2}]}{(j^2+\xi^2)^{\nu/2}} \, \mathrm{d}j$$
$$= \left(\frac{\partial J_\nu(x, y; \xi)}{\partial x}\right)_{y,\xi}, \tag{A1}$$

where

$$J_\nu(x, y; \xi) = \int_0^\infty \cosh(jx) \frac{K_\nu[y(j^2+\xi^2)^{1/2}]}{(j^2+\xi^2)^{\nu/2}} \, \mathrm{d}j. \tag{A2}$$

Using the expansion

$$\cosh(jx) = \sum_{m=0}^{\infty} (jx)^{2m}/(2m)!,$$

the integral (Gradshteyn and Ryzhik 1965)

$$\int_0^\infty j^{2\mu+1} \frac{K_\nu[\alpha(j^2+z^2)^{1/2}]}{(j^2+z^2)^{\nu/2}} \, \mathrm{d}j = \frac{2^\mu \Gamma(\mu+1)}{\alpha^{\mu+1} z^{\nu-\mu-1}} K_{\nu-\mu-1}(\alpha z),$$

and the relation

$$\Gamma(m+\tfrac{1}{2}) = \pi^{1/2} \frac{(2m)!}{2^{2m} m!},$$

we obtain

$$J_\nu(x, y; \xi) = \left(\frac{\pi}{2y}\right)^{1/2} \sum_{m=0}^{\infty} \frac{(x^2/2y)^m}{m!} \frac{K_{\nu-m-1/2}(y\xi)}{\xi^{\nu-m-1/2}}. \tag{A3}$$

The summation over m appearing here can be carried out exactly by using the remarkable formula (Erdélyi 1953)

$$\sum_{m=0}^{\infty} \frac{(\frac{1}{2}z\tau^2)^m}{m!} \frac{K_{\lambda+m}(zs)}{s^{\lambda+m}} = \frac{K_\lambda[z(s^2-\tau^2)^{1/2}]}{(s^2-\tau^2)^{\lambda/2}},$$

remembering at the same time that $K_{-\nu}(z) = K_\nu(z)$. It follows that

$$J_\nu(x, y; \xi) = \left(\frac{\pi\xi}{2}\right)^{1/2} \frac{(y^2-x^2)^{\nu/2-1/4}}{(\xi y)^\nu} K_{\nu-1/2}[\xi(y^2-x^2)^{1/2}]. \tag{A4}$$

A partial differentiation with respect to x then yields the desired result

$$I_\nu(x, y; \xi) = \left(\frac{\pi\xi^3}{2}\right)^{1/2} \frac{x(y^2-x^2)^{\nu/2-3/4}}{(\xi y)^\nu} K_{\nu-3/2}[\xi(y^2-x^2)^{1/2}]. \tag{A5}$$

We may as well write down some other integrals which are needed for the evaluation of the specific heat of the system, namely

$$L_\nu(x, y; \xi) = \int_0^\infty j^2 \cosh(jx) \frac{K_\nu[y(j^2+\xi^2)^{1/2}]}{(j^2+\xi^2)^{\nu/2}} \, dj$$
$$= \left(\frac{\partial I_\nu(x, y; \xi)}{\partial x}\right)_{y,\xi}$$
$$= \left(\frac{\pi\xi^3}{2}\right)^{1/2} \frac{1}{(\xi y)^\nu} \{(y^2-x^2)^{\nu/2-3/4} K_{\nu-3/2}[\xi(y^2-x^2)^{1/2}]$$
$$+ \xi x^2 (y^2-x^2)^{\nu/2-5/4} K_{\nu-5/2}[\xi(y^2-x^2)^{1/2}]\}, \tag{A6}$$

whence one obtains another desired result, namely

$$-J_\nu(x, y; \xi) + y L_{\nu+1}(x, y; \xi)$$
$$= \left(\frac{\pi\xi^3}{2}\right)^{1/2} \frac{x^2}{(\xi y)^\nu} (y^2-x^2)^{\nu/2-3/4} K_{\nu-3/2}[\xi(y^2-x^2)^{1/2}]$$
$$= x I_\nu(x, y; \xi). \tag{A7}$$

It seems important to point out here that if the hyperbolic functions appearing in the above integrals are replaced by exponentials the resulting integrals turn out to be intractable! This might explain why the previous authors, who did not include antiparticles into the problem, were unable to make much progress with the relativistic case.

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